BRIDGE NUMBER AND CONWAY PRODUCTS

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Abstract. Schubert proved that, given a composite link K with summands K_1 and K_2, the bridge number of K satisfies the following equation:

$$\beta(K) = \beta(K_1) + \beta(K_2) - 1.$$

In "Conway Produts and Links with Multiple Bridge Surfaces", Scharlemann and Tomova proved that, given links K_1 and K_2, there is a Conway product $K_1 \times_c K_2$ such that

$$\beta(K_1 \times_c K_2) \leq \beta(K_1) + \beta(K_2) - 1$$

In this paper, we define the generalized Conway product $K_1 *_c K_2$ and prove the lower bound $\beta(K_1 *_c K_2) \geq \beta(K_1) - 1$ where K_1 is the distinguished factor of the generalized product. We go on to show this lower bound is tight for an infinite class of links with arbitrarily high bridge number.

Introduction

Bridge number was introduced by Schubert in his paper "Uber eine Numerische Knoteninvariante." Here Schubert proves that, given a composite knot K with summands K_1 and K_2, the bridge number of K satisfies the following equation:

$$\beta(K) = \beta(K_1) + \beta(K_2) - 1.$$

The techniques used in this paper are inspired by Schultens' more modern proof of the same equality [5].

In this paper K, will be a tame link embedded in S^3 and $h : S^3 \to \mathbb{R}$ is a height function with level sets consisting of 2-spheres and two exceptional points corresponding to $+\infty$ and $-\infty$. We require that h restricts to a Morse function on K.

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Definition 1. *If the maxima of $h|_K$ occur above all of the minima then K is in bridge position. The fewest number of maxima of $h|_K$ over all embeddings of K is the bridge number of K, denoted $\beta(K)$.*

Definition 2. *A sphere C embedded in S^3 which meets a link K transversely in four points is called a Conway sphere.*

Let $K_1 \subset S^3_1$ and $K_2 \subset S^3_2$ be links embedded in distinct 3-spheres. For each $i = 1, 2$ let τ_i be arcs in S^3_i such that $\partial\tau_i \subset K_i$ but τ_i is otherwise disjoint from K_i. Let $\eta(\tau_i)$ be a regular closed neighborhood of τ_i, then $\eta(\tau_i) \cap K_i$ is a trivial tangle and $\partial(\eta(\tau_i))$ is a Conway sphere for K_i. Let $B_i = S^3_i - int(\eta(\tau_i))$.

Definition 3. *Let $K_1 *_c K_2$ (the* ***generalized Conway product*** *of K_1 and K_2) denote the link in S^3 formed by removing $int(\eta(\tau_i))$ from S^3_i and gluing $\partial(B_1)$ to $\partial(B_2)$ via a homeomorphism which sends $K_1 \cap \partial(B_1)$ to $K_2 \cap \partial(B_2)$).*

The image C of $\partial(\eta(\tau_1))$ and $\partial(\eta(\tau_2))$ after their identification is the Conway sphere of the generalized Conway product.

We call $K_1 *_c K_2$ a rational completion of K_1 if $(B_2, K_2 \cap B_2)$ is a rational tangle.

It is also important to note that the link type of $K_1 *_c K_2$ is dependent on K_1, K_2, τ_1, τ_2, and the gluing homeomorphism.

Note that nowhere do we require that the Conway sphere in the a generalized Conway product be incompressible. If the Conway sphere is compressible and $K_1 *_c K_2$ is prime, then one of the factor links is a 1 or 2 bridge link. For a further discussion of this special case, see Example 1.

The classical Conway sum and Conway product were originally defined in [1] as operations which received as input two tangle diagrams and produced as output a new tangle diagram. This original operation has inspired several related constructions. In [2], Lickorish studies a method of producing prime links by identifying together the boundaries of prime tangles. Scharlemann's and Tomova's operation takes two links, evacuates untangles from the links' complements to form two tangles, and identifies together the boundaries of these two tangles to form a new link[3]. The definition of generalized Conway product used in this paper encapsulates the construction in [3]. By carefully choosing τ_1, τ_2 and the gluing map, Scharlemann and Tomova showed the existence of a generalized Conway product which respects bridge surfaces. They go on to prove that the following inequality holds for such a product

$$\beta(K_1 *_c K_2) \leq \beta(K_1) + \beta(K_2) - 1$$

However, it is also shown in [3](via a construction by the author) that the above inequality is not always an equality, so a lower bound is needed.

The main goal of this paper is to present a lower bound on the bridge number of the generalized Conway product in terms of the bridge number of the factor links.

Theorem A. (Main Theorem) Let $K_1 *_c K_2$ be a generalized Conway product and K_1 be the distinguished factor, then

$$\beta(K_1 *_c K_2) \geq \beta(K_1) - 1$$

In addition, there is an infinite family of generalized Conway products with arbitrarily high bridge number for which $\beta(K_1 *_c K_2) = \beta(K_1) - 1$.

The term "distinguished factor" which appears in the above theorem will be defined later in the paper.

I am grateful to Martin Scharlemann for suggesting that I investigate the relationship between Conway products and bridge number and for many helpful conversations.

Conway Spheres

This section is devoted to generalizing work of Schultens [5] on companion tori in link complements to the case of Conway spheres.

For the remainder of the paper, K will be the generalized Conway product $K_1 *_C K_2$ embedded in S^3 with Conway sphere C.

We adopt the following notation from [3]. A **(punctured) disk** will denote a disk embedded in S^3 which is disjoint from K or meets K transversely in a single point. A simple closed curve in a Conway sphere C is **c-inessential** if it bounds a (punctured) disk in C.

Definition 4. *Let $\digamma_C$ be the singular foliation on the Conway sphere C induced by $h|_C$. Let σ be a leaf corresponding to a saddle singularity (by general position we can assume every such σ is disjoint from K). Then σ consists of two circles s_1 and s_2 wedged at a point. If either s_1 or s_2 is c-inessential in C, then we say σ is a c-inessential saddle. Otherwise, σ is c-essential.*

The following lemma and its proof are immediate generalizations of Schultens' Lemma 1 [5]. We need alter the statement and proof only slightly to account for punctures in the Conway sphere.

Lemma 1. *Let h, K, $\digamma_C$, C be as above. If $\digamma_C$ contains c-inessential saddles then after an isotopy of C that does not change the number of*

maxima of $h|_K$ there is a c-inessential saddle σ for which the following properties hold:

1) s_1 bounds a (punctured) disk $D_1 \subset C$ such that $\digamma_C$ restricted to D_1 contains only disjoint circles and one maximum or minimum.

2) For L the level sphere of h containing σ, D_1 co-bounds a 3-ball B with a disk $\tilde{D} \subset L - s_1$, such that B does not contain $+\infty$ or $-\infty$, and such that s_2 does not meet B.

Proof: Choose a c-inessential saddle $\sigma = s_1 \vee s_2$ to be innermost in C. Up to relabeling, s_1 bounds a (punctured) disk $D_1 \subset C$ satisfying the first property. s_1 cuts the level sphere L containing σ into two disks $\tilde{D_1}$ and $\tilde{D_2}$. $D_1 \cup \tilde{D_1}$ and $D_2 \cup \tilde{D_2}$ bound 3-balls $\tilde{B_1}$ and $\tilde{B_2}$ respectively. Up to relabeling, $\tilde{B_1}$ contains $+\infty$ or $-\infty$ and $\tilde{B_2}$ contains neither. If s_2 does not meet $\tilde{B_2}$ then property 2 is satisfied and we are done.

Suppose $s_2 \subset \tilde{D_2} \subset \tilde{B_2}$. Let us assume D_1 contains a single maximum p and $\tilde{B_1}$ contains $+\infty$ (the other situation is proved analogously). By general position, we can assume $h|_C$ does not have local maxima or minima at $K \cap C$. Choose α to be a monotone arc with end points p and $+\infty$ which intersects C only at local maxima. Label the points of $C \cap \alpha$ starting at p and increasing toward $+\infty$ as $p, p_1, p_2, ..., p_n$. See Fig. 1. Let S_+ be a level sphere contained in a small neighborhood of $+\infty$ such that S_+ does not meet C or K. Let β_n be a subarc of α with endpoints p_n and $+\infty$. Enlarge β_n slightly to be a vertical solid cylinder V such that $\partial(V)$ consists of a small disk in D_1 a small disk in S_+ and an annulus A with $\digamma_A$ a collection of circles. Replacing C with the Conway sphere $(C - V) \cup A \cup (S_+ - V)$ represents an isotopy of C in $S^3 - K$ which does not change the number of minima or maxima of $h|_C$.

By induction on n, we can assume α is disjoint from C except at the point p. By isotopying D_1 to a new disk D_1^* in the manner described above, we have enlarged $\tilde{B_2}$ to contain $+\infty$ and shrunk $\tilde{B_1}$ so that it is disjoint from $+\infty$. After a small tilt so that h again restricts to a Morse function on D_1^*, $\digamma_{D_1^*}$ is a collection of circles and one maximum. By choosing D_1^*, $\tilde{B_1}$, and $\tilde{D_1}$ to be D_1, B, and $\tilde{D}$ respectively we achieve property 2. $\square$

Definition 5. *Following* [5], *say a Conway sphere C is taut with respect to $\beta(K)$ if the number of saddles of $\digamma_C$ is minimal subject to the condition that $h|_K$ has $\beta(K)$ maxima.*

Lemma 2. *Let h, K, C, $\digamma_C$ be as above. If C is taut with respect to $\beta(K)$, then there are no c-inessential saddles in $\digamma_C$.*

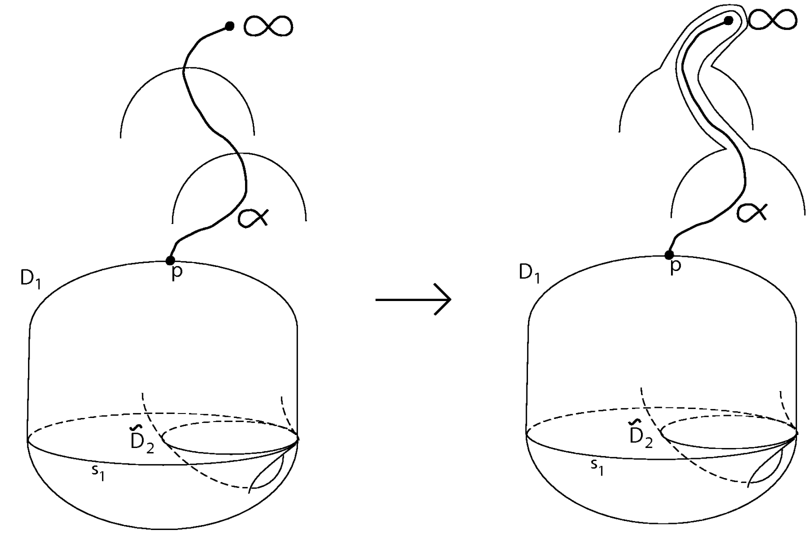

FIGURE 1.

Proof: Suppose there is a c-inessential saddle. We can assume there exists a c-inessential saddle σ in $\mathcal{F}_C$ satisfying the conclusions of Lemma 1. Up to relabeling, s_1 bounds a (punctured) disk $D_1 \subset C$. If D_1 is a 0-punctured disk, then the conclusion follows from Schultens Lemma 2 [5].

Assume D_1 is a 1-punctured disk containing a single maximum p and lying above L, the level sphere containing σ(the other possible situation, a reflection through L, is proved analogously). Let $k = K \cap D_1$ and γ be the strand of $K \cap B$ that contains k as a endpoint. The following isotopy was originally described on page 5 of [5].

If γ is monotone with respect to h or the closest critical point on γ is a minimum, we can skip ahead to the isotopy described in the next paragraph. Otherwise, let r be the maximum of $h|_\gamma$ closest to k along γ. Let α be a monotone arc contained in B starting at r and ending at p, the maximum of D_1. Let β be an arc in D_1 transverse to $\mathcal{F}_C$ starting at k and ending at p. α together with β and γ' (the segment of γ connecting k to r) bound a disk E with interior contained in B. K intersects E in γ' and transversely in points $q_1, ..., q_n$. let q_i be the highest such point of intersection. Let $\rho \subset (K \cap B)$ be the arc containing q_i and τ a small monotone sub-arc of ρ containing q_i. Replace τ with a monotone arc which starts at an end point of τ, runs parallel to E until it nearly reaches D_1, travels along D_1 until it returns to the other side of E, travels parallel to E (now on the opposite side) and connects to the other end point of τ. The result is isotopic to

K, does not change the number of maxima of $h|_K$ and reduces n. By induction on n, we may assume that $K \cap E = \gamma'$. Isotope γ' along E until it lies just outside of D_1 except where it intersects D_1 exactly at the point p. This isotopy of K does not change the number or nature of the maxima of $h|_K$. See Fig. 2.

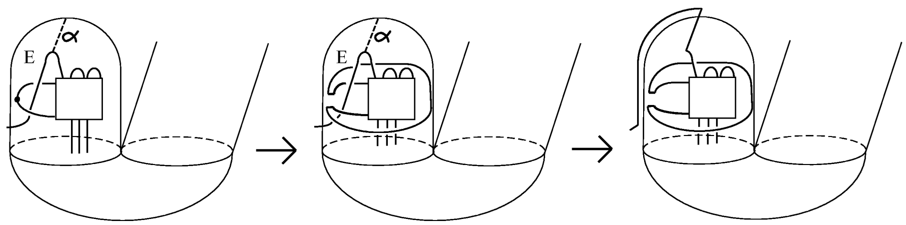

Figure 2.

At this point $(K \cup C) \cap int(B)$ can be shrunk horizontally and lowered to lie just below $\tilde{D}$. This isotopy produces a monotone arc connecting p to the image of $K \cap int(B)$ under the isotopy and does not change the number or nature of critical points of $h|_C$ or $h|_K$.

Since $D_1 \cup \tilde{D}$ bounds a ball minus an unknotted arc, we can isotope D_1 to $\tilde{D}$ to create $\tilde{C}$. After a small tilt, we have produced a new Conway sphere $\tilde{C}$ which is isotopic to C while preserving the number of maxima of $h|_K$. See Fig. 3. Since the number of saddles of $\mathcal{F}_{\tilde{C}}$ is one less than the number of saddles of $\mathcal{F}_C$, we have a contradiction to the assumption that C is taut.□

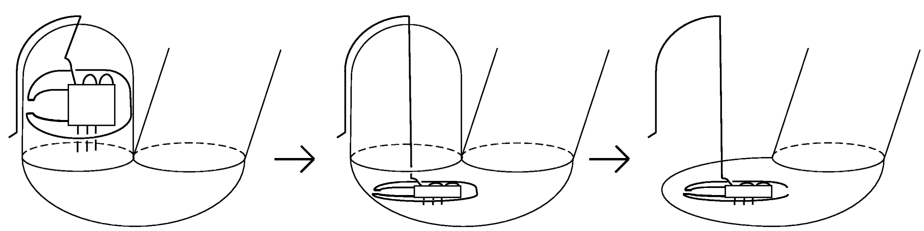

Figure 3.

Let σ be a saddle in $\mathcal{F}_C$. The bicollared neighborhood of σ in C has three boundary components c_1, c_2, and c_3 where c_1 and c_2 are parallel to s_1 and s_2 respectively. By the above lemma, if C is taut then neither c_1 nor c_2 bound (punctured) disks. Since C is a 4-punctured sphere, both c_1 and c_2 bound twice-punctured disks to each side. Consequently, c_3 bounds a disk to one side and a 4-punctured disk to the other. Thus, the saddles of a taut Conway sphere are stacked as illustrated in Fig. 4.

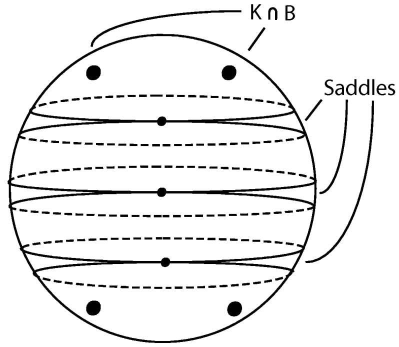

FIGURE 4.

C decomposes S^3 into two 3-balls B_1 and B_2. We may assume c_1 and c_2 are contained in the same level surface L. $L-(c_1\cup c_2)$ is composed of two disks and an annulus A. If a collar of $\partial(A)$ in A is contained in B_1, then we say σ is unnested with respect to B_1. If not, we say σ is nested with respect to B_1. We define nested and unnested with respect to B_2 similarly. Note that nested with respect to B_1 is the same as unnested with respect to B_2 and nested with respect to B_2 is unnested with respect to B_1.

Lemma 3. *Let h, K, C, $\mathcal{F}_C$ be as above. If C is taut with respect to $\beta(K)$, then all saddles of $\mathcal{F}_C$ are nested with respect to the same B_i, $i=1,2$.*

Proof: Suppose σ_1 and σ_2 are a saddles of $\mathcal{F}$ such that σ_i is nested with respect to B_i for $i=1,2$. We can assume σ_1 and σ_2 are adjacent in C. If σ_1 is the circles s_1^1 and s_2^1 wedged at a point and σ_2 is the circles s_1^2 and s_2^2 wedged at a point, then, up to labeling, s_1^1 and s_1^2 co-bound an annulus in C which is disjoint form all other saddles and does not meet K. Here we invoke Schultens' Lemma 3 where she constructs an isotopy of C which eliminates one saddle of $\mathcal{F}_C$ while preserving the number of maxima and minima of $h|_K$. This contradicts the tautness of C. $\square$

Summerizing the previous lemmas: if C is taut with respect to $\beta(K)$, then we may assume all saddles of $\mathcal{F}_C$ are c-essential and nested with respect to B_1 (up to labeling). At this point, B_1 can be visualized as a neighborhood of a knotted arc embedded in S^3. This useful embedding of B_1 allows us to bound $\beta(K)$ in terms of $\beta(K_1)$. Hence, we call K_1 the **distinguished factor**. It is relevant to note that B_1 and B_2 are simultaneously realized as neighborhoods of knotted arcs iff $\mathcal{F}_C$ contains no saddles.

Inequalities

Let $\{\sigma_1, \sigma_2, ..., \sigma_n\}$ be the set of saddles in $\mathcal{F}_C$. If C is taut, then let D_1 and D_2 be the two twice punctured disks in $C - \bigcup_{i=1}^{n} \sigma_i$. We use the following labeling convention: $\{x_1^i, x_2^i\} = K \cap D_i$ and $h(x_1^i) \geq h(x_2^i)$ for $i = 1, 2$. We will want to keep track of the following properties:

1)Is x_j^i a local minimum or maximum of $h|_{K\cap B_1}$ for $i = 1, 2$ and $j = 1, 2$?

2)Does $h|_{D_i}$ have a unique local minimum or maximum for $i = 1, 2$? (i.e. Is D_i a cap or a cup?)

To accomplish this we define a 3-tuple labeling $(x, y, z)\epsilon\{m, M\}^3$ for each D_i where where $x = m$ (resp. M) if x_1^i is a minimum (resp. maximum) of $h|_{K\cap B_1}$, where $y = m$ (resp. M) if x_2^i is a minimum (resp. maximum) of $h|_{K\cap B_1}$, and $z = m$ (resp. M) if $h|_{D_i}$ has a unique local minimum (resp. maximum).

As an example, the disk in Fig. 5 is labeled (M, m, m).

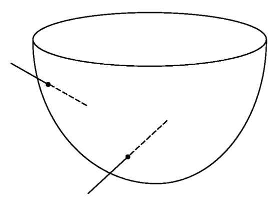

Figure 5.

Lemma 4. *Given h, K, C, $\mathcal{F}_C$ as above. There is an isotopy of K preserving the number of maxima of $h|_K$ and resulting in $h|_K$ having at least one maximum or minimum in B_2.*

Proof: We assume C is taut. If $\mathcal{F}_C$ contains saddles then D_1 and D_2 are defined as in the above discussion. If $\mathcal{F}_C$ has no saddles then let s be a level curve in $\mathcal{F}_C$ which separates two points in $C \cap K$ from two others. The two components of $C - s$ are the twice-punctured disks D_1 and D_2.

We will proceed by cases using the 3-tuple labeling of D_1 and D_2. An underscore in a coordinate of a labeling will indicate m or M.(i.e. $(m, \_, M)$ represents (m, m, M) or (m, M, M)).

Claim: Neither D_1 nor D_2 is labeled (M, M, M) or (m, m, m).

Suppose to get a contradiction that D_1 is labeled (M, M, M).Let $\partial(D_1) = s_1$ and σ be the saddle in $\mathcal{F}_C$ containing s_1. Let L be the level surface containing σ. Let $\{x_1, x_2\} = \{x_1^i, x_2^i\} = K \cap D_1$.

By appealing to the proof of Lemma 1, we assume D_1 co-bounds a 3-ball B with a disk $\tilde{D} \subset L - C$, such that B does not contain $+\infty$ or $-\infty$, and such that s_2 does not meet B.

$K \cap int(B)$ can be shrunk horizontally and lowered to lie just below $\tilde{D}$. This isotopy produces two monotone arcs in B connecting x_1 and x_2 to the image of $K \cap int(B)$ under the isotopy and does not change the number or nature of critical points of $h|_C$ or $h|_K$.

Since $D_1 \cup \tilde{D}$ bounds a ball minus two monotone unknotted arcs, we can isotope D_1 to $\tilde{D}$ to create $\tilde{C}$. After a small tilt, we have produced a new Conway sphere $\tilde{C}$ which is isotopic to C while preserving the number of maxima of $h|_K$. See Fig. 7. Since the number of saddles of $\mathcal{F}_{\tilde{C}}$ is one less than that of $\mathcal{F}_C$, we have a contradiction to the assumption that C is taut. The other possibilities in this case are proved analogously.

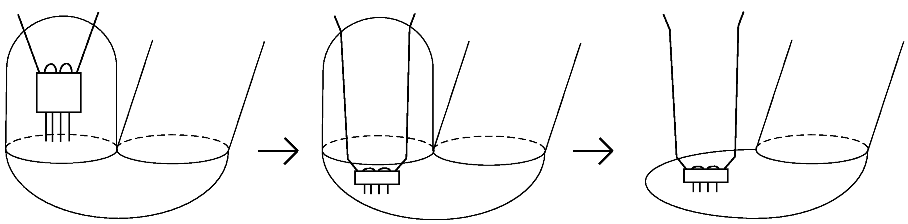

Figure 6.

Case 1: One of D_i for $i = 1, 2$ is labeled $(m, \_, M)$ or $(\_, M, m)$.

Up to renaming of the disks, let D_1 have the 3-tuple label $(m, \_, M)$. Let $\partial(D_1) = s_1$ and σ be the saddle in $\mathcal{F}_C$ containing s_1. Let L be the level surface containing σ. Let $\{x_1, x_2\} = \{x_1^i, x_2^i\}$ and γ be the strand of $K \cap B_1$ that contains x_1 as an endpoint, so γ ascends from x_1 into B_1.

By appealing to the proof of Lemma 6, we assume D_1 co-bounds a 3-ball B with a disk $\tilde{D} \subset L - C$, such that B does not contain $+\infty$ or $-\infty$, and such that s_2 does not meet B.

We proceed as in the proof of Lemma 8. Let r be the maximum of $h|_\gamma$ closest to x_1 along γ. Let α be a monotone arc contained in B starting at r and ending at p, the maximum of D_1. Let β be an arc in D_1 transverse to $\mathcal{F}_C$ starting at x_1 and ending at p. α together with β and γ' (the segment of γ connecting x_1 to r) bound a disk E with interior contained in B. K intersects E in γ' and transversely in points $q_1, ..., q_n$. let q_i be the highest such point of intersection. Let $\rho \subset (K \cap B)$ be the arc containing q_i and τ a small monotone sub-arc of ρ containing q_i. Replace τ with a monotone arc which starts at an

end point of τ runs parallel to E until it nearly reaches D_1, travels along D_1 until it returns to the other side of E, travels parallel to E (now on the opposite side) and connects to the other end point of τ. Since $h(x_1) \geq h(x_2)$, then $h(q_i) \geq h(x_2)$ for $i = 1, ..., n$ and the link resulting from the above arc replacement is isotopic to K. See Fig. 6. As in Lemma 8, this isotopy does not change the number of maxima of $h|_K$ but does reduce $n = |K \cap int(E)|$. By induction on n, we may assume that $K \cap E = \gamma'$. Isotope γ' along E until it lies just out side of D_1 except where it intersects D_1 exactly at the point p. Again, this isotopy of K does not change the number of maxima of $h|_K$ nor does it alter the tautness of C. We conclude $h|_K$ has at least one maximum in B_2. The proof if D_i is labeled $(m, \_, M)$ is analogous.

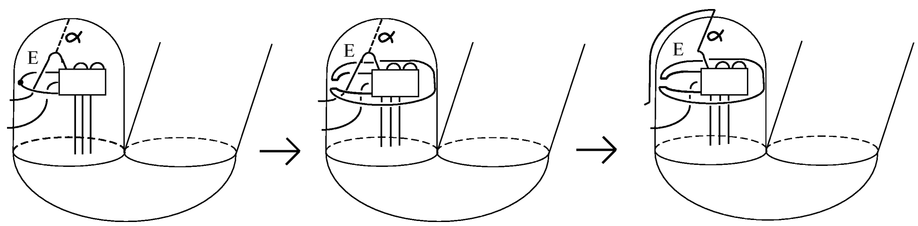

FIGURE 7.

Case 2: The labels of D_1 and D_2 are both chosen from the set $\{(M, m, m), (M, m, M)\}$.

The disks corresponding to these two possible labelings are depicted in Fig. 8.

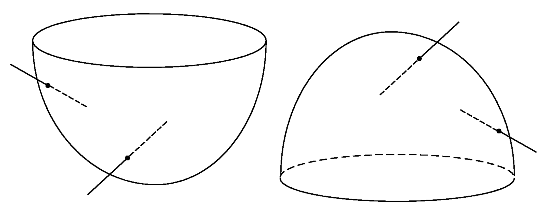

FIGURE 8.

Suppose D_1 is labeled (M, m, M) and D_2 is labeled (M, m, m). Let α be the component of $K \cap B_2$ with an end point x_1^1. If α contains a maximum or minimum of $h|_K$, then we are done. If not, then α is monotone and the other endpoint of α must be x_2^2. This leaves x_2^1 and x_1^2 connected by β, the other component of $K \cap B_2$. The monotonicity of α ensures $h(x_2^2) \geq h(x_1^1)$. Since $h(x_1^1) \geq h(x_2^1)$, $h(x_1^2) \geq h(x_2^2)$ and $h(x_2^2) \geq h(x_1^1)$, then $h(x_2^1) \geq h(x_1^2)$. However, x_2^1 is labeled M and x_1^2 is labeled m, so there must be both a minimum and a maximum of $h|_K$

in $\beta \subset B_2$. See Fig 9. This result follows analogously for the other possible labelings of D_1 and D_2.$\square$

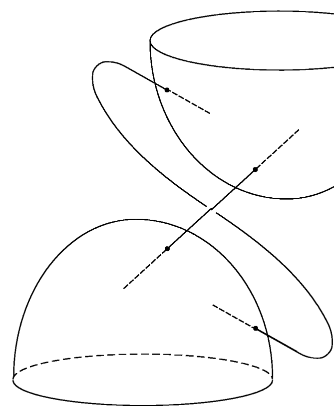

FIGURE 9.

Theorem A. Let h, K, C, $\mathcal{F}_C$ be as above. Then the following inequality holds:

$$\beta(K) \geq \beta(K_1) - 1$$

Where K_1 is the distinguished factor.

Proof: By the previous lemmas, we can assume that C has no inessential saddles, C is nested with respect to B_2, and $h|_K$ has at least one maximum in B_2(the case where $h|_K$ has one minimum in B_2 is proved analogously). To prove the theorem, we need only prove that the number of maxima of $h|_K$ in B_1 is greater than or equal to $\beta(K_1)-2$. The theorem will then follow since $\beta(K) =$ (number of maxima of $h|_K$ in B_1)+(number of maxima of $h|_K$ in B_2) $\geq \beta(K_1)-2+1 = \beta(K_1)-1$.

First, we analyze the case where $\mathcal{F}_C$ contains no saddles. If C contains no saddles, there is a level preserving isotopy of S^3 taking C to a standard round 2-sphere. Such an isotopy preserves the number and nature of maxima of $h|_K$ in B_1. As in Lemma 10, a point in $K \cap C$ is labeled with an m if it is a local minimum of $h|_{K \cap B_1}$ and is labeled with an M if it is a local maximum of $h|_{K \cap B_1}$. The link K_1 can be recovered from $K \cap B_1$ by taking a rational completion of K_1 using a rational tangle T. If more points of $K \cap C$ are labeled with an M, take T to lie above C. If more are labeled with an m, take T to lie below C. See Fig. 10. Since the portion of the rational tangle lying in the region labeled R can be taken to be monotone with respect to h, this rational completion causes the creation of at most two new maxima. The number of maxima of the resulting embedding of K_1 is at most

two more than the number of maxima of $h|_K$ in B_1. Hence, the number of maxima of $h|_K$ in B_1 is greater than or equal to $\beta(K_1) - 2$.

(Note: If $\mathcal{F}_C$ has no saddles, we get the analogous estimate that the number of maxima of $h|_K$ in B_2 is greater than or equal to $\beta(K_2) - 2$. Hence, in this special case, we get the additional inequality $\beta(K) \geq \beta(K_1) + \beta(K_2) - 4$.)

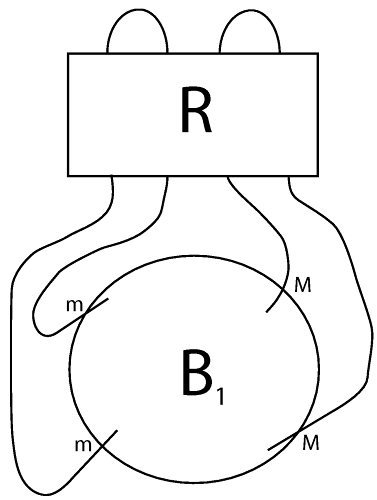

Figure 10.

We now assume $\mathcal{F}_C$ contains saddles. To establish the desired inequality in this general setting, we build an isotopy of S^3 which takes B_1 to a standard round 3-ball and preserves the number and nature of critical points of $h|_K$ in B_1. This isotopy, however, does not preserve the number of critical points of $h|_K$ in B_2. Let D_1 be one of the twice punctured disks in $C - \bigcup_{i=1}^{n} \sigma_i$. Let $\partial(\bar{D}_1) = s_1$ and σ be the saddle in $\mathcal{F}_C$ containing s_1. $\mathcal{F}_{D_1}$ is a collection of circles and one point corresponding to a maximum of $h|_C$(if the point is a minimum, the case is analogous). Let L be the level surface containing σ and $\tilde{D}$ be the disk component of $L - s_1$ which does not meet s_2. D_1 and $\tilde{D}$ co-bound a 3-ball B. By appealing to the proofs of Lemma 6, we can assume B does not meet $+\infty$. Let $x_1, x_2 = K \cap D_1$. Each point x_i receives a label as described above. Since $h|_{D_1}$ has a maximum as the unique critical point, we can horizontally shrink and vertically lower $B \cup D_1$ until D_1 lies just above $\tilde{D}$. Let D_1^* be the image of D_1 under this isotopy and let p be the unique maximum of $h|_{D_1^*}$. Let J be the level surface containing p. By general position, $J \cap C$ consists of the point p and a collection of circles. One such circle c_2 is parallel in C to s_2. By picking D_1^* close enough to $\tilde{D}$, we can choose a point b in c_2 and an arc α in J which

is disjoint from C except at its boundary $\{b, p\}$. Choose another arc β in C which does not meet K, has boundary $\{b, p\}$ and is transverse to $\mathcal{F}_C$ everywhere accept where it passes through $s_1 \cap s_2$. Having made D_1^* sufficiently close to $\tilde{D}$ we can assume α and β co-bound a disk F which is vertical with respect to h, disjoint from K, and disjoint from C except along β. Isotope C along F to effectively cancel a saddle with a maximum. See Fig. 11.

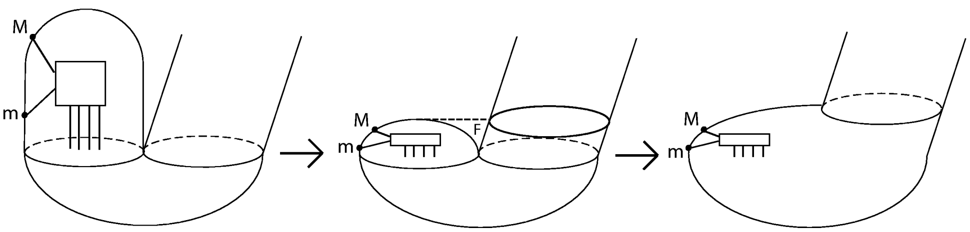

Figure 11.

Repeat this process to produce an isotopy $\Phi : S^3 \to S^3$ so that $\mathcal{F}_{\Phi(C)}$ contains no saddles. By the previous argument, $\mathcal{F}_{\Phi(C)}$ has no saddles implies the number of maxima of $h|_{\Phi(K)}$ in $\Phi(B_1)$ is greater than or equal to $\beta(K_1) - 2$. However, Φ was constructed so that the number of maxima of $h|_{\Phi(K)}$ in $\Phi(B_1)$ is equal to the number of maxima of $h|_K$ in B_1. Hence, the number of maxima of $h|_K$ in B_1 is greater than or equal to $\beta(K_1) - 2$. This completes the proof of the theorem.□

Examples

Example 1

It is important to note that nowhere in the proof of Theorem 1 do we need incompressibility of our Conway sphere C. One might ask how we can reconcile Theorem 1 with the fact that there exist rational completions of the unknot with arbitrarily high bridge number. In fact, any Whitehead double of a knot is an example of such a link. In such cases, the distinguished factor is always a rational link. See Fig. 12. Hence, K_1 has bridge number at most 2. If we now employ Theorem 1, we get the following trivial inequality $\beta(K_1 *_c K_2) \geq \beta(K_1) - 1 \geq 2 - 1 \geq 1$.

Example 2

In Fig. 13, K_1 is the connect sum of four trefoils and K_2 is a satellite link with a trefoil as companion. Schuebert's seminal work on bridge number tells us that $\beta(K_1) = 5$ and $\beta(K_2) \geq 4$ [4]. Since Fig. 13

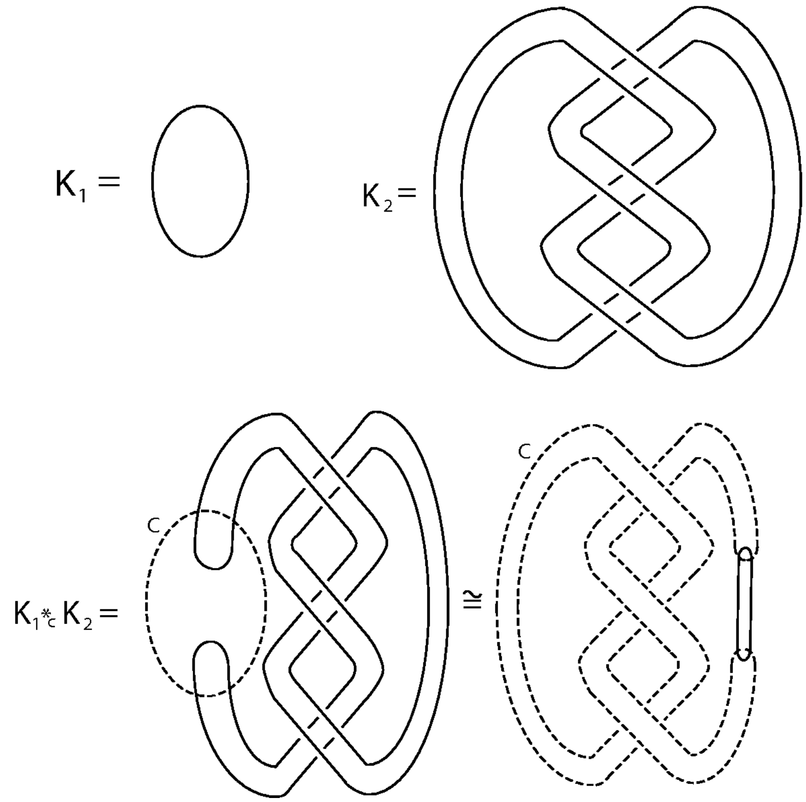

FIGURE 12.

gives a presentation of K_2 with exactly 4 maxima we conclude that $\beta(K_2) = 4$. The link $K = K_1 *_c K_2$ depicted in Fig. 5 is a satellite link also with a trefoil as companion. Again Schubert's results tell us that $\beta(K) \geq 4$ and again we have a presentation of K with exactly 4 maxima. Hence, $\beta(K) = 4 = \beta(K_1) - 1$.

To extend this particular example to an infinite family of links where $\beta(K) = \beta(K_1) - 1$ simply take K_2 to be a $(p, 2)$ cable link with an n-bridge knot as companion and K_1 to be the connect sum of $2n$ copies of a 2-bridge link. After a construction analogous to that in Fig. 13, $K_1 *_c K_2$ will be a satellite link with bridge number $2n$. Hence, $\beta(K_1 *_c K_2) = 2n = (2n + 1) - 1 = \beta(K_1) - 1$. We conclude that the bound given in the main theorem is tight for an infinite family of generalized Conway products with arbitrarily high bridge number.

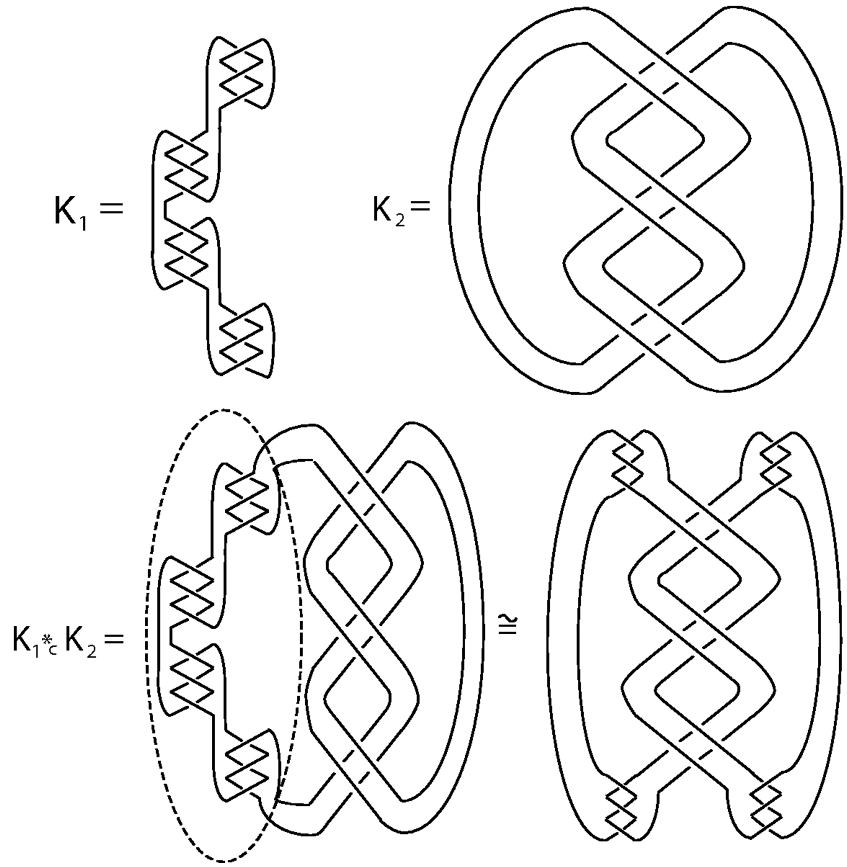

Figure 13.

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